Name SOLUTIONS

ECE 147B

Midterm Exam

Feb. 12, 2020

- You are allowed one, single-sided 8 ½ x 11 sheet of self-prepared notes.
- For full credit, ALWAYS SHOW WORK to explain your answers.
- If/when stuck, write what you know for partial credit; then MOVE ON!
- **No calculators, laptops, or other computing devices are allowed.**

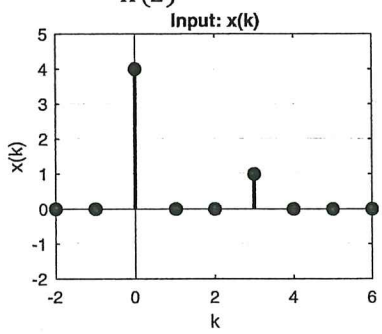
Good Luck!!

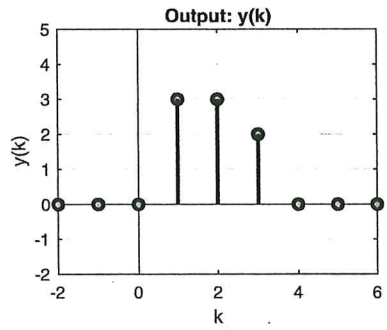
/25	**Problem 1:**	**Discrete-time transfer functions**
/21	**Problem 2:**	**Impulse and step responses**
/24	**Problem 3:**	**Emulation – s plane vs z plane poles**
/30	**Problem 4:**	**State space equations**

/100 **TOTAL**

Problem 1. Discrete-time transfer functions.

Below are an observed input, x(k) and corresponding output, y(k), for some transfer function $G(z) = \frac{Y(z)}{X(z)}$. Note that x(k)=y(k)=0 for all other values of k that are not shown.





$$Y(z) = \frac{3z^2+3z+2}{z^3}$$

$$X(z) = \frac{4z^3+1}{z^3}$$

a) Solve for G(z).

$$G(z) = \frac{Y(z)}{X(z)} = \frac{3z^2+3z+2}{4z^3+1}$$

b) Write a difference equation corresponding to G(z).

$$4y_{k+3} + y_k = 3x_{k+2} + 3x_{k+1} + 2x_k$$

$$y_k = \frac{-1}{4}y_{k-3} + \frac{3}{4}x_{k-1} + \frac{3}{4}x_{k-2} + \frac{1}{2}x_{k-3}$$

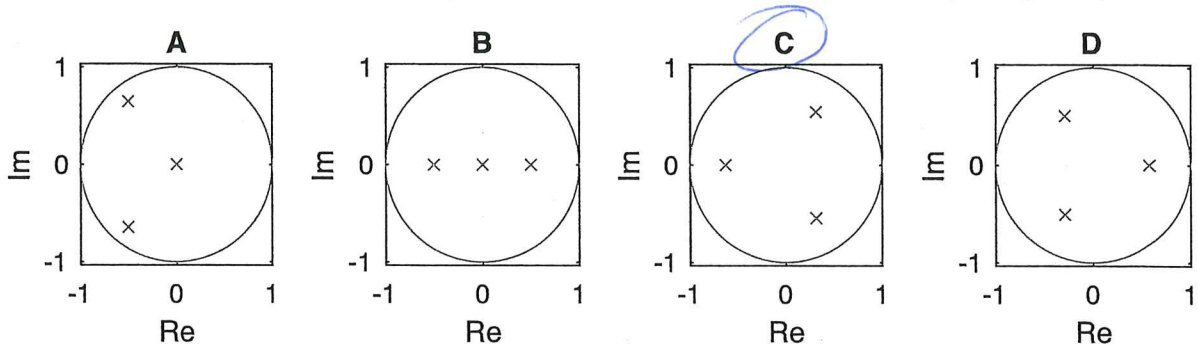
c) Solve for the first 4 values of a unit step response of G(z), i.e., for k=0 through k=3.

(Note: this can certainly be done by using your solution to (b), but, if you are uncertain of that answer, it can also be done by looking directly at the input and output, using superposition.)

k	x_k	y_k
0	1	0
1	1	$0.75 = \frac{3}{4}$
2	1	$1.5 = \frac{3}{4} + \frac{3}{4}$
3	1	$2.0 = \frac{3}{4} + \frac{3}{4} + \frac{1}{2}$
4	1	$1\frac{13}{16} = \left(\frac{3}{4} + \frac{3}{4} + \frac{1}{2}\right) - \frac{1}{4}\cdot\frac{3}{4}$

d) Which (if any) of the plots below shows the locations of the POLES of G(z)? Explain, in terms of r and theta. (The zeros are intentionally NOT shown here – only the poles.)

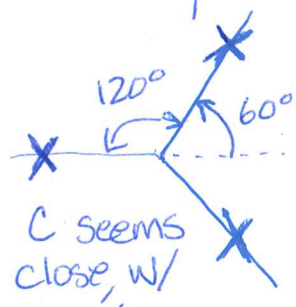
Ans: C



Poles are solutions to the characteristic eqn:

$4z^3 + 1 = 0$

$\boxed{z^3 = -0.25}$ → 3 solutions, spaced by 120°, each with $r = \sqrt[3]{.25}$ →

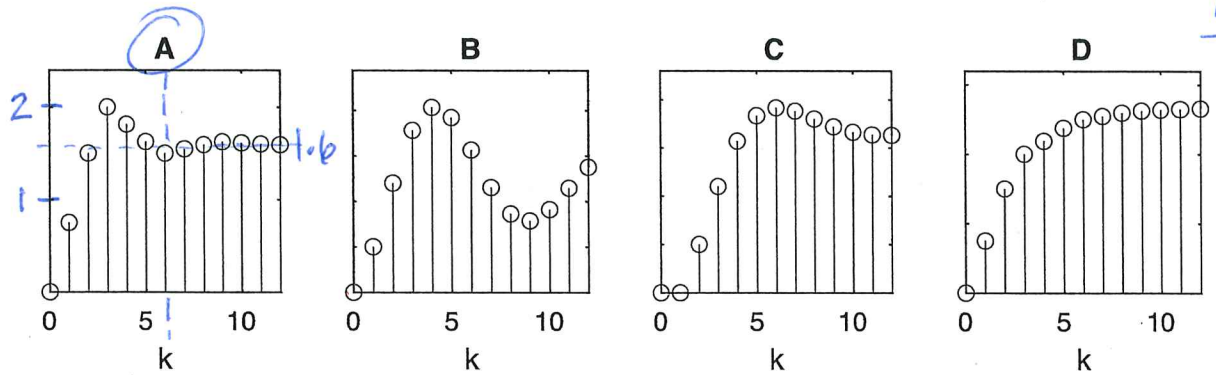


$\frac{360°}{60°} = 6$ pts per cycle.

C seems close, w/ $r \approx .6$ or so

e) Which (if any) of the plots below shows a unit step response for G(z)? Explain. (For partial credit, explain why certain plots can be "eliminated" from consideration.)

Ans: A

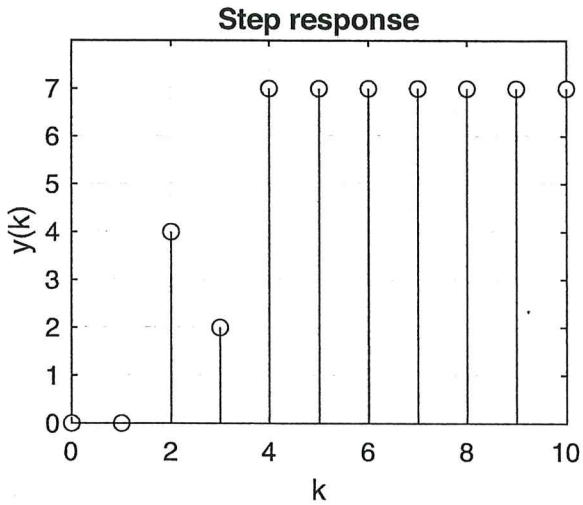


Period-6, and first 4 values of y_k match part c.

Final value should be $\frac{3+3+2}{4+1} = \frac{8}{5} = 1.6$, which looks like A.

Problem 2. Impulse and step responses.

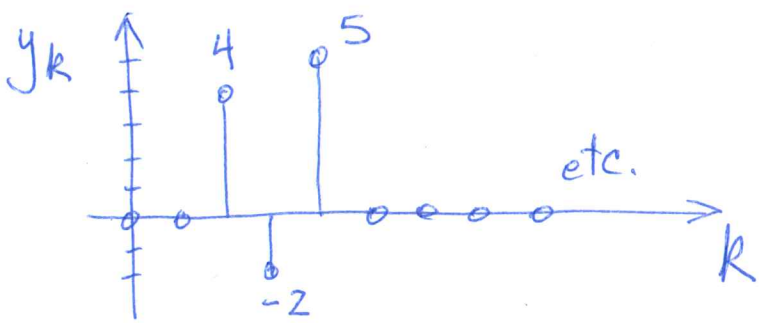
Below is a unit step response for some plant, P(z).



Subtraction a copy, delayed by one time step:

0	0	4	2	7	7
0	0	0	-4	-2	-7
0	0	4	-2	5	0

a) Sketch a unit impulse response for the plant.

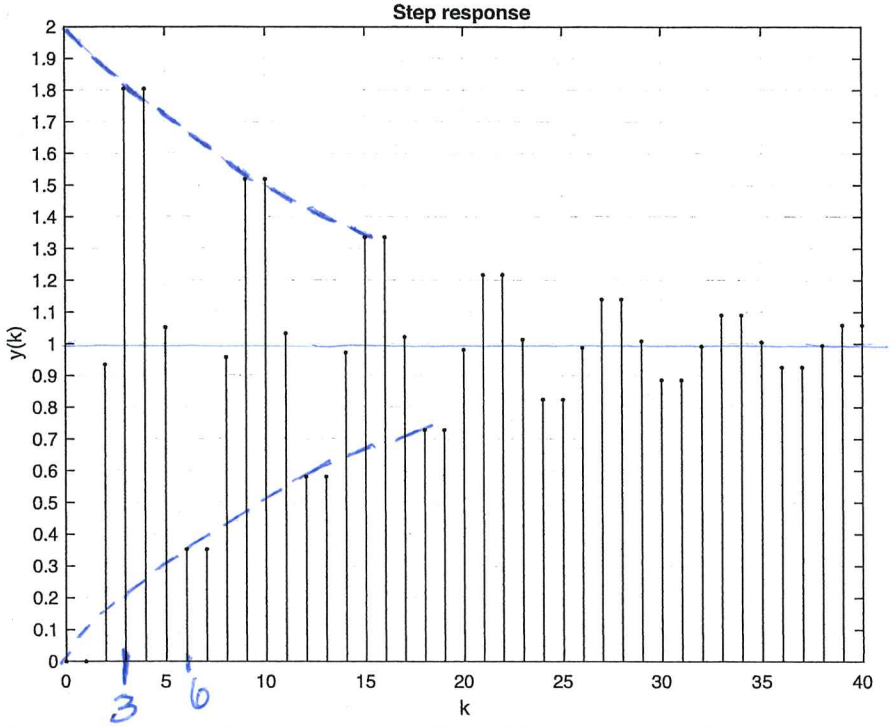


b) Solve for the transfer function, P(z).

If x_k is a unit impulse, $X(z)=1$.

So, $P(z) = Y(z) = 4z^{-2} - 2z^{-3} + 5z^{-4} = \dfrac{4z^2 - 2z + 5}{z^4}$

Below is the step response of a second-order plant, G(z). (Second-order means it has two poles.)



← No zero(s), since $y(k=0) = y(k=1) = 0$.

c) Estimate r and theta, numerically, of the corresponding z plane pole pair.

At $k=3$, decay goes from $(2-1)$ to $(1.8-1)$, approx.

So, $r^3 \approx \frac{(1.8-1)}{(2-1)} = 0.8 = 1 - 0.2$

$r = (1-0.2)^{1/3} \approx 1 - \frac{1}{3} \cdot 0.2 \approx 0.93$

$\theta = \frac{360^\circ}{6} = 60^\circ$ or $\theta = \frac{2\pi}{6} = \frac{\pi}{3}$ (rad), since 6 pts per cycle.

d) Estimate G(z), numerically, given the final value of the step response shown is 1. (Hint: Does G(z) have any zeros, and why?)

Denominator of $G(z)$ is: $z^2 - 2r\cos\theta\, z + r^2$

$\cos(60^\circ) = 0.5$

$r \approx 0.93$

$\approx z^2 - 2(0.93)(\frac{1}{2})z + (.93)^2$

$\approx z^2 - 0.93z + 0.8649$

$(1-.07)^2$

$= 1 - 2 \cdot .07 + .0049$

$= .8649$

Numerator: $1 - 0.93 + 0.8649$

$= .07 + 0.8649 = 0.9349$

$$G(z) \approx \frac{0.9349}{z^2 - 0.93z + 0.8649}$$

For full credit, you MUST SHOW WORK to estimate the z plane poles for each case. There is no need to solve numerically for the location, so long as you have provided equations or other information/reasoning sufficient to choose among the six options provided.

a) pole-zero matching (aka 'matched') E

$r \approx 0.1$ to 0.15...

$z = e^{sT} = \exp(-2 \pm 7j)$. $r = e^{-2}$, & $\theta = 7$ (rad)

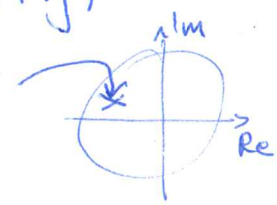
So, this θ "wraps" past 2π, due to aliasing.

$\theta - 2\pi \approx 7 - 6.3 \approx 0.7$ (rad), so 40° or 45° or so.

b) Tustin F

$$z_1 = \frac{(2+sT)}{(2-sT)} = \frac{(2-2+7j)}{(2+2-7j)} = \frac{7j}{(4-7j)} \frac{(4+7j)}{(4+7j)}$$

$$z_1 = \frac{-49+28j}{16+49} = \frac{-49+28j}{65}$$, which is around here.



c) Forward Euler D

$z = 1 + sT$

$= 1 - 2 \pm 7j$

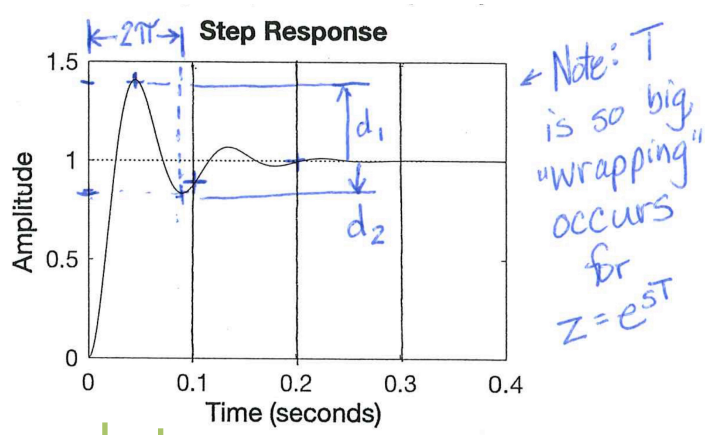
$= -1 \pm 7j$, which is far outside unit circle: D

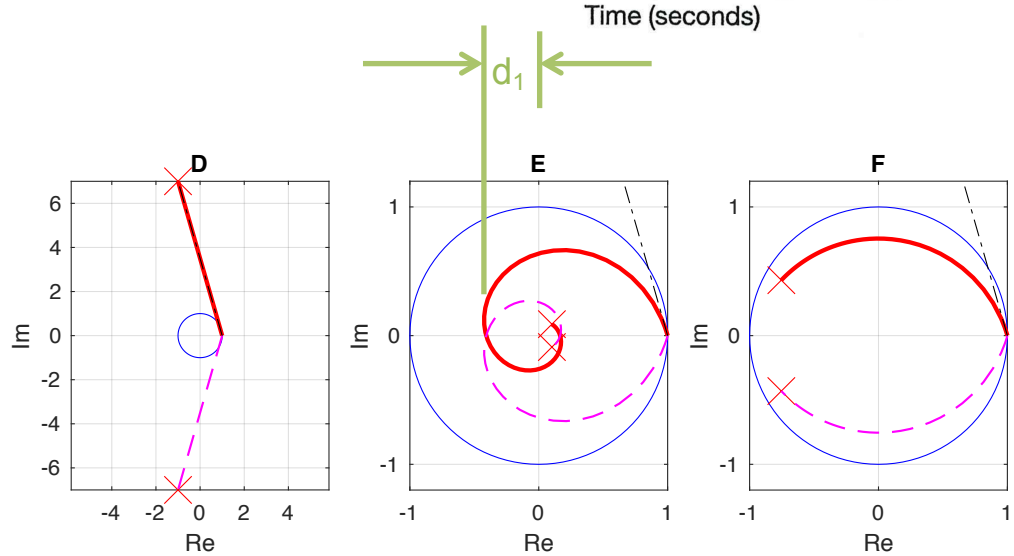
d) On each of the 3 plots you chose, sketch the locus of values of the z plane poles for all values of T between 0 and 0.1 seconds. (This is equivalent to taking the straight line between 0 and s=-20+70j on the s plane, and sketching a curve approximating its conformal mapping location on the z plane.) You need not be precise. The next page shows an example of such curves corresponding to the backward Euler case, for clarity.

Problem 3. Emulation – s plane vs z plane poles

Note: the step response gives information about the pole-zero matching transformation (z=exp(s*T)). For one thing, T=0.1 (s) is so large that it corresponds to "wrapping" by more than 2*pi on the z plane. Also, the peak value, d1, corresponds to the most negative real value of the spiral trajectory for plot "E".

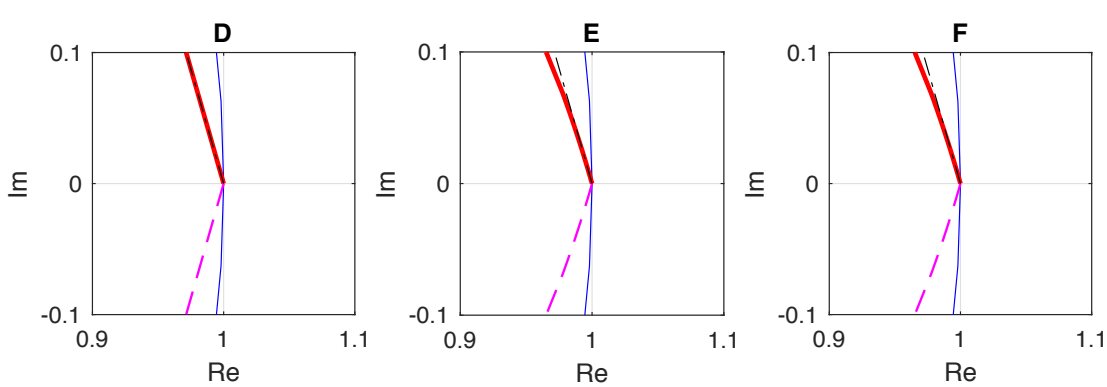
$s = -20 \pm 70j$





Above: the trajectories as T goes from 0 to 0.1 seconds.

Below: Note that all three transformations look similar in the limit as T gets very small.



Problem 4. State space equations.

Below is a CONTINUOUS-TIME state space system, $sX = AX + BU$, with output defined as: $Y = CX + DU$.

$$A = \begin{bmatrix} -8 & -25 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 2 & 4 \end{bmatrix}, D = [0] .$$

a) Calculate the transfer function from input, u, to output, y: $\frac{Y(s)}{U(s)} = ?$

$sX_1 = -8X_1 - 25X_2 + 2u$

$sX_2 = X_1$

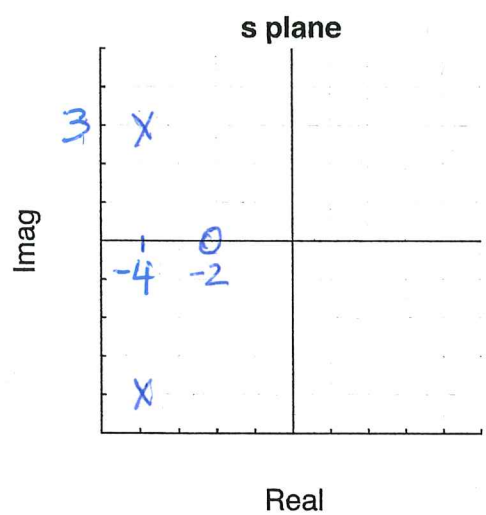
$s^2X_2 + 8sX_2 + 25X_2 = 2u \rightarrow \frac{X_2}{u} = \frac{2}{s^2+8s+25}$

$Y = 2X_1 + 4X_2$

$\frac{X_1}{u} = \frac{sX_2}{u} = \frac{2s}{s^2+8s+25}$

$\frac{Y}{u} = 2\frac{X_1}{u} + 4\frac{X_2}{u} = \frac{4s+8}{s^2+8s+25}$

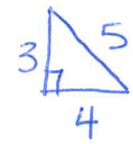
b) Calculate the s-plane poles and zero(s) of the system, and sketch them on the axes below:



zero: $4(s+2)=0$, $\therefore s=-2$

poles: $s^2+8s+25=0$

$s=-4\pm 3j$



Note: the point of sketching the s plane poles is to create a mental image of z plane poles. Since forward Euler just involves scaling and shifting the pole and zero locations, one should expect the plot for part "d)" to look scaled (by 0.25), and translated such that s=0 maps to z=1.

(Choose appropriate scaling for axes.)

Problem 4. (continued...)

$T = \frac{1}{4}(s)$

c) Convert the A and B matrices to a DISCRETE-TIME (DT) state space system using the forward Euler approximation method for a sample time of T=0.25 seconds. (The DT C and D matrices will remain unchanged.) Recall the system given earlier was:

$A = \begin{bmatrix} -8 & -25 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 4 \end{bmatrix}$, $D = [0]$. Assume the states are x_1 and x_2: $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

$$A_d = I + AT = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -2, & -\frac{25}{4} \\ .25, & 0 \end{bmatrix} = \begin{bmatrix} -1, & -6.25 \\ .25, & 1 \end{bmatrix}$$

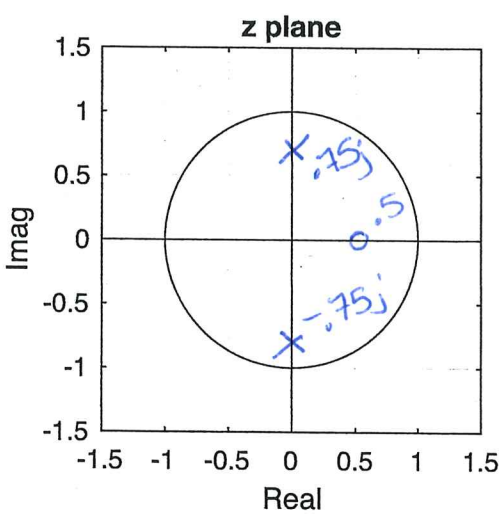
$$B_d = BT = \begin{bmatrix} 0.25 \\ 0 \end{bmatrix}$$

Note: B was misprinted on this page and was meant to match B from the last page.

The solution at left corresponds to B as listed on this page, but a solution with "0.5" instead of "0.25" is also correct, given this typo.

We have emphasized in class that transforming a CT dynamic system using emulation (e.g., forward Euler) results in the same DT system, whether we convert a CT TF to a DT TF or whether we convert state space matrices.

d) Calculate the DT poles and zero(s) of your new system, and sketch them on the axes below:



$$Z = 1 + sT$$

pole: $= 1 + \frac{(-4 \pm 3j)}{4}$

$= 1 - 1 \pm \frac{3}{4}j = \pm 0.75j$

zero: $Z = 1 + sT$

$= 1 - \frac{2}{4}$

$= 0.5$

Problem 4. (continued...)

e) For the discrete-time system, describe and briefly explain the following characteristics of a unit step response. As usual, this means the response of output y, given input u is a unit step, and assuming zero initial conditions for the system.

- Does it have oscillations? Yes,

- If so, estimate/calculate the period of oscillation. 4 step per cycle, & $T = \frac{1}{4}$ $\therefore$ period is 1 sec

- Is it stable? Yes, since $r < 1$.

- If so, what is the final value to a unit step input?

Same as for $\frac{Y(s)}{U(s)}$:

$$\lim_{s \to 0} \left(s \cdot \frac{1}{s} \cdot \frac{Y(s)}{U(s)} \right) = \frac{8}{25} = 0.32$$

Using a the (typo) version of B, which is off by a factor of two, would yield 0.16 here.